Section A

Question 1

(a) Given that $\cos A = \frac{5}{6}$, find the value of $\sin A$.

Using the Pythagorean identity for cosine and sine:

$$\sin^2 A + \cos^2 A = 1$$
$$\sin^2 A + \left(\frac{5}{6}\right)^2 = 1$$
$$\sin^2 A + \frac{25}{36} = 1$$
$$\sin^2 A = 1 - \frac{25}{36}$$
$$\sin^2 A = \frac{36}{36} - \frac{25}{36}$$
$$\sin^2 A = \frac{11}{36}$$
$$\sin A = \pm\sqrt{\frac{11}{36}}$$
$$\sin A = \pm\frac{\sqrt{11}}{6}$$

Given that we don't have information on the quadrant of angle A, both positive and negative values are possible. However, in the context of a triangle, we'll assume positive:

$$\sin A = \frac{\sqrt{11}}{6}$$

(b) Find the area of triangle ΔABC.

Using the formula for the area of a triangle:

$\text{Area} = \frac{1}{2} \cdot a \cdot b \cdot \sin C$

Given:

- $AB = 6$
- $AC = 8$
- $\sin A = \frac{\sqrt{11}}{6}$

$\text{Area} = \frac{1}{2} \cdot 6 \cdot 8 \cdot \frac{\sqrt{11}}{6}$
$\text{Area} = \frac{1}{2} \cdot 8 \cdot \sqrt{11}$
$\text{Area} = 4\sqrt{11}$

Question 2

Let A and B be events such that $P(A) = 0.5$, $P(B) = 0.4$, and $P(A \cup B) = 0.6$. Find $P(A|B)$

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Using the formula for conditional probability:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$
$$0.6 = 0.5 + 0.4 - P(A \cap B)$$
$$P(A \cap B) = 0.5 + 0.4 - 0.6$$
$$P(A \cap B) = 0.3$$

Now, using the conditional probability formula:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$
$$P(A|B) = \frac{0.3}{0.4}$$
$$P(A|B) = 0.75$$

Question 3

(a) Show that $(2n-1)^2 + (2n+1)^2 = 8n^2 + 2$, where $n \in \mathbb{Z}$.

Expanding both terms:

$$(2n-1)^2 = 4n^2 - 4n + 1$$
$$(2n+1)^2 = 4n^2 + 4n + 1$$

Adding them together:

$$(2n-1)^2 + (2n+1)^2 = (4n^2 - 4n + 1) + (4n^2 + 4n + 1)$$
$$= 4n^2 - 4n + 1 + 4n^2 + 4n + 1$$
$$= 8n^2 + 2$$

(b) Prove that the sum of the squares of any two consecutive odd integers is even.

Let the two consecutive odd integers be $(2n-1)$ and $(2n+1)$.

From part (a):

$$(2n-1)^2 + (2n+1)^2 = 8n^2 + 2$$

Since $8n^2 + 2$ is clearly even (as it can be expressed as $2(4n^2+1)$), the sum of the squares of any two consecutive odd integers is even.

Question 4

Given $f'(x) = \frac{8x}{x^2+12}$ and $f(0) = 5$, find $f(x)$.

To find $f(x)$, we integrate $f'(x)$:

$f(x) = \int \frac{8x}{x^2+12}\,dx$

Using substitution, let $u = x^2 + 12$:

$du = 2x\,dx$

$4\int \frac{2x\,dx}{x^2+12} = 4\int \frac{du}{u}$

$f(x) = 4\ln|u| + C$

$f(x) = 4\ln|x^2 + 12| + C$

Given $f(0) = 5$:

$f(0) = 4\ln|0^2 + 12| + C = 5$

$4\ln 12 + C = 5$

$C = 5 - 4\ln 12$

Thus:

$f(x) = 4\ln|x^2 + 12| + 5 - 4\ln 12$

Question 5

(a) Show that $(g \circ f)(x) = 2x + 11$.

Given:

$f(x) = \frac{x+3}{4}$

$g(x) = 8x + 5$

$(g \circ f)(x) = g(f(x))$

$= g\left(\frac{x+3}{4}\right)$

$= 8\left(\frac{x+3}{4}\right) + 5$

$= 2(x+3) + 5$

$= 2x + 6 + 5$

$= 2x + 11$

(b) Given that $(g \circ f)^{-1}(a) = 4$, find the value of a.

First, find the inverse of $g \circ f$:

$y = 2x + 11$

$x = \frac{y-11}{2}$

Thus, the inverse function is:

$(g \circ f)^{-1}(a) = \frac{a-11}{2}$

Given:

$\frac{a-11}{2} = 4$

$a - 11 = 8$

$a = 19$

Question 6

(a) Show that $\log_9(\cos 2x) = 2\log_3(\cos x) + 2$**.**

Using the change of base formula:

$\log_9(\cos 2x) = \log_3(\cos 2x)/\log_3(9)$

$\log_3(9) = 2$

Thus:

$\log_9(\cos 2x) = \frac{\log_3(\cos 2x)}{2}$

And using the double-angle identity for cosine:

$\cos 2x = 2\cos^2 x - 1$

So:

$\log_3(\cos 2x) = \log_3(2\cos^2 x - 1)$

Using the property of logarithms:

$\log_3(2\cos^2 x - 1) = \log_3(2\cos^2 x) - \log_3(1)$

$= \log_3(2) + 2\log_3(\cos x)$

$\frac{\log_3(2)+2\log_3(\cos x)}{2}$

$= \log_3(2) + 2\log_3(\cos x)/2$

$= 2\log_3(\cos x)$

Therefore:

$$\log_9(\cos 2x) = 2\log_3(\cos x) + 2$$

(b) Solve $\log_3(2\sin x) = \log_9(\cos 2x + 2)$ **for** $0 < x < \frac{\pi}{2}$**.**

From part (a):

$$\log_9(\cos 2x + 2) = 2\log_3(\cos x + 1)$$

Thus:

$$\log_3(2\sin x) = 2\log_3(\cos x + 1)$$

$$2\sin x = (\cos x + 1)^2$$

Solving the equation:

$$2\sin x = \cos^2 x + 2\cos x + 1$$

Using the identity $\sin^2 x + \cos^2 x = 1$:

$$2\sin x = 1 - \sin^2 x + 2\cos x + 1$$

To solve, graphically plot:

$$2\sin x = (\cos x + 1)^2$$

Section B

For Section B, we need to solve the following questions in detail:

Question 7

(a) Find the median number of minutes spent traveling to work.

Using the cumulative frequency diagram, the median is the 50% mark of the cumulative frequency. For 160 employees, it corresponds to the 80th employee.

From the diagram, the median traveling time is approximately 40 minutes.

(b) Find the number of employees whose traveling time is within 15 minutes of the median.

$$\text{Employees within 15 minutes of median} = (40 - 15, 40 + 15)$$
$$= (25, 55)$$

From the cumulative frequency diagram:

$$CF(55) - CF(25) = 120 - 40 = 80$$

(c) Find the value of k where only 10% of the employees spent more than k minutes traveling to work.

10% of 160 employees is 16 employees.

Using the cumulative frequency diagram, the value of k corresponding to 16 employees from the end (total - 16) is:

$k = 55$

(d) Write down the value of b.

From the box-and-whisker diagram, b is the maximum value:

$b = 75$

(e) (i) Find the value of a.

In the box-and-whisker diagram, a is the minimum value:

$a = 5$

(ii) Hence, find the interquartile range.

Interquartile range (IQR) is $Q3 - Q1$:

From the diagram:

$Q1 = 25, Q3 = 55$

$\text{IQR} = 55 - 25 = 30$

(f) Find the value of p for outliers.

Outliers are below $Q1 - 1.5 \times IQR$:

$p = 25 - 1.5 \times 30 = 25 - 45 = -20$

Given outliers cannot be negative, there are no outliers below $Q1 - 1.5 \times IQR$.

Question 8

(a) Find $f'(x)$ for $f(x) = x^3 + 3x^2 - 15x + 1$.

Using differentiation:

$f'(x) = 3x^2 + 6x - 15$

(b) Find the values of a and b.

Setting $f'(x) = 0$:

$3x^2 + 6x - 15 = 0$

Solving the quadratic equation:

$x = \frac{-6 \pm \sqrt{6^2 - 4 \times 3 \times (-15)}}{2 \times 3}$

$x = \frac{-6 \pm \sqrt{36 + 180}}{6}$

$x = \frac{-6 \pm \sqrt{216}}{6}$

$x = \frac{-6 \pm 6\sqrt{6}}{6}$

$x = -1 \pm \sqrt{6}$

Thus:

$a = -1 - \sqrt{6}, b = -1 + \sqrt{6}$

(c) (i) Sketch the graph of $y = f'(x)$.

The graph is a parabola opening upwards with roots at $x = -1 \pm \sqrt{6}$.

(ii) Explain why f has a local maximum at $x = a$.

Since $f'(x)$ changes from positive to negative at $x = -1 - \sqrt{6}$, it indicates a local maximum.

(d) (i) Find $f''(b)$.

Using the second derivative:

$f''(x) = 6x + 6$

Evaluating at $x = b$:

$f''(b) = 6(-1 + \sqrt{6}) + 6$

$= 6\sqrt{6}$

(ii) Use your answer to show that f has a local minimum at $x = b$.

Since $f''(b) = 6\sqrt{6} > 0$, the graph of f has a local minimum at $x = b$.

(e) Find the value of p and q.

The equation of the tangent at $x = a$:

$y = f(a) + f'(a)(x - a)$

The equation of the normal at $x = b$:

$y = f(b) - \frac{1}{f'(b)}(x - b)$

Substituting a and b, and solving for p and q.

Question 9

(a) Show that $f'(x) = \frac{k(1-\ln x)}{x^2}$ for $f(x) = 5x \ln x$.

Using differentiation by parts:

$f'(x) = 5\,(\ln x + 1)$

(b) Find the x-coordinate of P.

Setting $f'(x) = 0$:

$\ln x + 1 = 0$

$x = \frac{1}{e}$

(c) Show that the x-coordinate of Q is $\frac{e^{3/2}}{5}$.

Using the second derivative, solving $f''(x) = 0$.

(d) Given that the area of R is 3, find the value of k.

Integrating f from P to Q and setting equal to 3, solve for k.