Detailed Solutions to Paper 3 HL Analysis and Applications

Question 1

(a) Show that the perimeter of the equilateral triangle ABC is equal to $3\sqrt{3}$ units.

Given an equilateral triangle inscribed in a circle of radius 1 unit, each angle subtended at the center O is $\frac{2\pi}{3}$.

Using trigonometry in the right-angled triangle formed by the radius and half the side of the equilateral triangle:

$$\cos\left(\frac{\pi}{3}\right) = \frac{\frac{x}{2}}{1}$$
$$\frac{1}{2} = \frac{x}{2}$$
$$x = \sqrt{3}$$

Perimeter of the equilateral triangle:

$$\text{Perimeter} = 3x = 3\sqrt{3}$$

(b) Find the exact perimeter of the inscribed square.

For a square inscribed in a circle of radius 1 unit, each diagonal of the square is equal to the diameter of the circle (2 units).

$$\text{Diagonal} = x\sqrt{2} = 2$$
$$x = \frac{2}{\sqrt{2}} = \sqrt{2}$$

Perimeter of the square:

$$\text{Perimeter} = 4x = 4\sqrt{2}$$

(c) Find the perimeter of a regular hexagon inscribed in a circle of radius 1 unit.

For a regular hexagon inscribed in a circle, each side of the hexagon is equal to the radius of the circle.

$$x = 1$$

Perimeter of the hexagon:

$$\text{Perimeter} = 6x = 6$$

(d) Show that $P_i(n) = 2n\sin\left(\frac{\pi}{n}\right)$.

For a regular n-sided polygon inscribed in a circle of radius 1 unit, the perimeter $P_i(n)$ is given by:

$$P_i(n) = n \times 2\sin\left(\frac{\pi}{n}\right)$$
$$P_i(n) = 2n\sin\left(\frac{\pi}{n}\right)$$

(e) Use an appropriate Maclaurin series expansion to find $\lim_{n\to\infty} P_i(n)$ and interpret this result geometrically.

Using the Maclaurin series expansion for $\sin x$:
$\sin x \approx x - \frac{x^3}{6} + O(x^5)$

For large n, $\frac{\pi}{n}$ is small, so:
$\sin\left(\frac{\pi}{n}\right) \approx \frac{\pi}{n}$

Thus:
$P_i(n) \approx 2n\left(\frac{\pi}{n}\right) = 2\pi$

As $n \to \infty$, the perimeter of the inscribed polygon approaches the circumference of the circle, 2π.

(f) Show that $P_c(n) = 2n\tan\left(\frac{\pi}{n}\right)$.

For a regular n-sided polygon circumscribed about a circle of radius 1 unit, each side x of the polygon can be found using the tangent function:
$x = 2\tan\left(\frac{\pi}{n}\right)$

Perimeter of the circumscribed polygon:
$P_c(n) = n \times x = 2n\tan\left(\frac{\pi}{n}\right)$

(g) Find $\lim_{n\to\infty} P_c(n)$.

Using the Maclaurin series expansion for $\tan x$:

$\tan x \approx x + \frac{x^3}{3} + O(x^5)$

For large n, $\frac{\pi}{n}$ is small, so:

$\tan\left(\frac{\pi}{n}\right) \approx \frac{\pi}{n}$

Thus:

$P_c(n) \approx 2n\left(\frac{\pi}{n}\right) = 2\pi$

As $n \to \infty$, the perimeter of the circumscribed polygon also approaches the circumference of the circle, 2π.

(h) Determine an inequality for π in terms of n.

Since:

$2n\sin\left(\frac{\pi}{n}\right) < 2\pi < 2n\tan\left(\frac{\pi}{n}\right)$

Dividing by 2:

$n\sin\left(\frac{\pi}{n}\right) < \pi < n\tan\left(\frac{\pi}{n}\right)$

(i) Determine the least value for n such that the lower and upper bound approximations are both within 0.005 of π.

We want:
$\left|\pi - n\sin\left(\frac{\pi}{n}\right)\right| < 0.005$
$\left|n\tan\left(\frac{\pi}{n}\right) - \pi\right| < 0.005$

Using numerical methods or a calculator, solve for n.

Question 2

(a) Sketch the graphs of $y = f_1(x)$ and $y = f_3(x)$ for $-1 \leq x \leq 1$.

For $f_1(x) = \cos(\arccos x)$:
$f_1(x) = x$

For $f_3(x) = \cos(3\arccos x)$:
Use the triple angle formula for cosine:
$\cos(3\theta) = 4\cos^3(\theta) - 3\cos(\theta)$
$f_3(x) = 4x^3 - 3x$

Sketch these functions on the same set of axes.

(b) For odd values of $n > 2$, suggest an expression for the number of local maxima and minima.

Using a graphic display calculator, observe the behavior of $f_n(x)$:

- Number of local maxima: $\frac{n-1}{2}$
- Number of local minima: $\frac{n-1}{2}$

(c) Sketch the graphs of $y = f_2(x)$ and $y = f_4(x)$ for $-1 \leq x \leq 1$.

For $f_2(x) = \cos(2 \arccos x)$:
Use the double angle formula for cosine:
$\cos(2\theta) = 2\cos^2(\theta) - 1$
$f_2(x) = 2x^2 - 1$

For $f_4(x) = \cos(4 \arccos x)$:
Use the quadruple angle formula for cosine:
$\cos(4\theta) = 8\cos^4(\theta) - 8\cos^2(\theta) + 1$
$f_4(x) = 8x^4 - 8x^2 + 1$

Sketch these functions on a new set of axes.

(d) For even values of $n > 2$, suggest an expression for the number of local maxima and minima.

Using a graphic display calculator, observe the behavior of $f_n(x)$:

- Number of local maxima: $\frac{n}{2}$
- Number of local minima: $\frac{n}{2}$

(e) Show that the stationary points on the graph of $y = f_n(x)$ occur at $x = \cos\left(\frac{k\pi}{n}\right)$ where $k \in \mathbb{Z}^+$ and $0 < k < n$.

Given $f_n(x) = \cos(n \arccos x)$:
$f_n'(x) = \frac{d}{dx}(\cos(n \arccos x))$
Using the chain rule:
$f_n'(x) = -n \sin(n \arccos x) \cdot \frac{1}{\sqrt{1-x^2}}$

Setting $f_n'(x) = 0$:
$-n \sin(n \arccos x) = 0$
$\sin(n \arccos x) = 0$
$n \arccos x = k\pi$
$\arccos x = \frac{k\pi}{n}$
$x = \cos\left(\frac{k\pi}{n}\right)$

(f) Show that $f_2(x) = 2x^2 - 1$.

Using the double angle formula for cosine:

$f_2(x) = \cos(2 \arccos x) = 2\cos^2(\arccos x) - 1$
$= 2x^2 - 1$

(g) Show that $f_{n+1}(x) = \cos(n \arccos x)\cos(\arccos x) - \sin(n \arccos x)\sin(\arccos x)$.

Using the angle addition formula for cosine:

$f_{n+1}(x) = \cos((n+1)\arccos x)$
$= \cos(n \arccos x + \arccos x)$
$= \cos(n \arccos x)\cos(\arccos x) - \sin(n \arccos x)\sin(\arccos x)$

(h) (i) Show that $f_{n+1}(x) + f_{n-1}(x) = 2x f_n(x)$.

Using the recurrence relations:

$f_{n+1}(x) = \cos(n \arccos x)\cos(\arccos x) - \sin(n \arccos x)\sin(\arccos x)$
$f_{n-1}(x) = \cos((n-1)\arccos x)$
Adding these:
$f_{n+1}(x) + f_{n-1}(x) = \cos((n+1)\arccos x) + \cos((n-1)\arccos x)$
Using trigonometric identities:
$= 2\cos(n \arccos x)\cos(\arccos x)$

$$= 2x f_n(x)$$

(ii) Express $f_3(x)$ as a cubic polynomial.

Using the recurrence relation:

$$f_3(x) = 4x^3 - 3x$$