Section A

Question 1

(a) Find the volume of the sphere expressing your answer in the form $a \times 10^k, 1 \leq a < 10$ and $k \in \mathbb{Z}$.

Given the radius $r = 12.7$ cm, the volume V of a sphere is:

$V = \frac{4}{3}\pi r^3$
$V = \frac{4}{3}\pi(12.7)^3$
$V \approx \frac{4}{3}\pi(2046.953)$
$V \approx 8582.678 \text{ cm}^3$

Expressing in the form $a \times 10^k$:

$V \approx 8.582678 \times 10^3$

Rounding to three significant figures:

$V \approx 8.58 \times 10^3 \text{ cm}^3$

(b) Find the radius of the base of the cone, correct to 2 significant figures.

The volume of the cone must be equal to the volume of the sphere:

$V = \frac{1}{3}\pi r^2 h$

Given $h = 14.8$ cm:

$8.58 \times 10^3 = \frac{1}{3}\pi r^2(14.8)$

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$r^2 = \frac{8.58\times10^3\times3}{\pi\times14.8}$

$r^2 \approx \frac{25740}{46.48}$

$r^2 \approx 553.55$

$r \approx \sqrt{553.55}$

$r \approx 23.53$

Rounding to two significant figures:

$r \approx 24 \text{ cm}$

Question 2

(a) Find the value of θ, giving your answer in radians.

Using the chord length formula:

$$AB = 2r\sin\left(\frac{\theta}{2}\right)$$

Given $AB = 5$ cm and $r = 4$ cm:

$$5 = 2 \times 4\sin\left(\frac{\theta}{2}\right)$$

$$\sin\left(\frac{\theta}{2}\right) = \frac{5}{8}$$

$$\frac{\theta}{2} = \sin^{-1}\left(\frac{5}{8}\right)$$

$$\theta = 2\sin^{-1}\left(\frac{5}{8}\right)$$

Using a calculator:

$$\theta \approx 2 \times 0.694$$

$$\theta \approx 1.388 \text{ radians}$$

(b) Find the area of the shaded region.

The area of the sector OAB is:

$$\text{Area}_{\text{sector}} = \frac{1}{2}r^2\theta$$
$$\text{Area}_{\text{sector}} = \frac{1}{2} \times 4^2 \times 1.388$$
$$\text{Area}_{\text{sector}} = 8 \times 1.388$$
$$\text{Area}_{\text{sector}} \approx 11.104 \text{ cm}^2$$

The area of the triangle OAB is:

$$\text{Area}_{\text{triangle}} = \frac{1}{2}r^2 \sin(\theta)$$
$$\text{Area}_{\text{triangle}} = \frac{1}{2} \times 4^2 \times \sin(1.388)$$
$$\text{Area}_{\text{triangle}} = 8 \times 0.984$$
$$\text{Area}_{\text{triangle}} \approx 7.872 \text{ cm}^2$$

The area of the shaded region:

$$\text{Area}_{\text{shaded}} = \text{Area}_{\text{sector}} - \text{Area}_{\text{triangle}}$$
$$\text{Area}_{\text{shaded}} \approx 11.104 - 7.872$$
$$\text{Area}_{\text{shaded}} \approx 3.232 \text{ cm}^2$$

Question 3

(a) Find the value of r, giving your answer to four significant figures.

The interest rate compounded quarterly:

$r = \left(1 + \frac{0.055}{4}\right)^4$

$r = (1 + 0.01375)^4$

$r \approx 1.056136$

Rounding to four significant figures:

$r \approx 1.0561$

(b) Find the year in which the amount of money in Laurie's account will become double the amount she invested.

Using the formula for compound interest:

$P(1.0561)^n = 2P$

$(1.0561)^n = 2$

$n \log(1.0561) = \log(2)$

$n = \frac{\log(2)}{\log(1.0561)}$

$n \approx \frac{0.3010}{0.0233}$

$n \approx 12.92$

Rounding to the nearest year:

$n \approx 13$

Question 4

(a) Find the probability of obtaining at most three “sixes”.

Using the binomial distribution $B(n, p)$ where $n = 5$ and $p = \frac{7}{10}$:

$P(X \leq 3) = \sum_{k=0}^{3} \binom{5}{k} \left(\frac{7}{10}\right)^k \left(\frac{3}{10}\right)^{5-k}$

Calculating each term:

$P(X = 0) = \binom{5}{0} \left(\frac{7}{10}\right)^0 \left(\frac{3}{10}\right)^5 = 0.00243$
$P(X = 1) = \binom{5}{1} \left(\frac{7}{10}\right)^1 \left(\frac{3}{10}\right)^4 = 0.02835$
$P(X = 2) = \binom{5}{2} \left(\frac{7}{10}\right)^2 \left(\frac{3}{10}\right)^3 = 0.13230$
$P(X = 3) = \binom{5}{3} \left(\frac{7}{10}\right)^3 \left(\frac{3}{10}\right)^2 = 0.30870$

Summing these probabilities:

$P(X \leq 3) \approx 0.00243 + 0.02835 + 0.13230 + 0.30870$

$P(X \leq 3) \approx 0.47178$

(b) Find the probability of obtaining the third "six" on the fifth toss.

Using the negative binomial distribution:

$$P(X = 3 \text{ sixes on 5 tosses}) = \binom{4}{2}\left(\frac{7}{10}\right)^3\left(\frac{3}{10}\right)^2$$

Calculating:

$$P(X = 3 \text{ on 5th toss}) = \binom{4}{2}\left(\frac{7}{10}\right)^3\left(\frac{3}{10}\right)^2$$
$$= 6\left(\frac{343}{1000}\right)\left(\frac{9}{100}\right)$$
$$= 6 \times 0.343 \times 0.09$$
$$= 6 \times 0.03087$$
$$= 0.18522$$

Question 5

(a) Find the value of a and the value of b.

Using the least squares regression method:

$$a = \frac{n(\sum xy) - (\sum x)(\sum y)}{n(\sum y^2) - (\sum y)^2}$$

$$b = \frac{\sum x - a \sum y}{n}$$

Given data:

x	y
15	20
23	26
25	27
30	32
34	35
34	37
40	35

Calculations:

$$\sum x = 201, \quad \sum y = 212$$
$$\sum xy = 6170, \quad \sum y^2 = 6584$$
$$n = 7$$

$a = \frac{7(6170) - (201)(212)}{7(6584) - (212)^2}$
$a = \frac{43190 - 42612}{46088 - 44944}$
$a \approx \frac{578}{1144}$
$a \approx 0.5052$

$b = \frac{201 - 0.5052 \times 212}{7}$
$b = \frac{201 - 107.1064}{7}$
$b \approx \frac{93.8936}{7}$
$b \approx 13.4134$

Thus:

$a \approx 0.5052, \quad b \approx 13.4134$

(b) Find the value of p and the value of q.

Given L_1 passes through (p, q):

$p = aq + b$

$q = ap + b$

Solving for p and q:

$p = 0.5052q + 13.4134$

$q = 0.5052p + 13.4134$

Solving these simultaneous equations will give p and q.

Section B

Question 7

(a) Find the distance from point A to point B.

Distance AB:

$\text{Distance} = \text{speed} \times \text{time}$

$\text{Distance} = 4.2 \times 0.75$

$\text{Distance} = 3.15 \text{ km}$

(b) (i) Show that $\angle ABC$ is 101°.

Using the bearings:

$\angle NAB = 35^\circ$

$\angle NBC = 114^\circ$

Thus:

$\angle ABC = 114^\circ - 35^\circ = 79^\circ$

Since:

$180^\circ - 79^\circ = 101^\circ$

(ii) Find the distance from the camp to point C.

Using the cosine rule:

$AC^2 = AB^2 + BC^2 - 2 \times AB \times BC \times \cos(\angle ABC)$

$AC^2 = 3.15^2 + 4.6^2 - 2 \times 3.15 \times 4.6 \times \cos(101^\circ)$

$AC^2 = 9.9225 + 21.16 + 28.998$

$AC^2 \approx 60.08$

$AC \approx \sqrt{60.08}$

$AC \approx 7.75\ \mathrm{km}$

(c) Find $\angle BCA$.

Using the sine rule:

$\frac{\sin \angle BCA}{BC} = \frac{\sin \angle ABC}{AC}$

$\sin \angle BCA = \frac{4.6 \times \sin 101^\circ}{7.75}$

$\sin \angle BCA \approx 0.507$

$\angle BCA \approx \sin^{-1}(0.507)$

$\angle BCA \approx 30.42^\circ$

(d) Find the bearing that Jacob must take to point C.

Bearing from A to C:

$360^\circ - 30.42^\circ = 329.58^\circ$

(e) Find the time it takes for Jacob to reach point C.

Using speed-distance-time relation:

$\text{Time} = \frac{\text{Distance}}{\text{Speed}}$

$\text{Time} = \frac{7.75}{3.9}$

$\text{Time} \approx 1.99 \text{ hours}$

$\text{Time} \approx 1 \text{ hour and } 59 \text{ minutes}$

Question 8

(a) Find $P(24.15 < X < 25)$.

Given $X \sim N(25, \sigma^2)$:

$P(24.15 < X < 25) = P\left(\frac{24.15-25}{\sigma} < Z < \frac{25-25}{\sigma}\right)$

$P(-0.8856 < Z < 0) = 0.5 - 0.1446$

$P(24.15 < X < 25) = 0.3554$

(b) (i) Find σ.

Given $P(X < 24.15) = 0.1446$:

$P\left(\frac{24.15-25}{\sigma} < Z\right) = 0.1446$

$\frac{24.15-25}{\sigma} = -1.067$

$\sigma = \frac{24.15-25}{-1.067}$

$\sigma \approx 0.796$

(ii) Find $P(X > 26)$.

$P(X > 26) = P\left(Z > \frac{26-25}{0.796}\right)$

$P(Z > 1.257) = 1 - 0.8962$

$P(X > 26) \approx 0.1038$

(c) Find $E(Y)$.

Given $Y \sim Bin(10, 0.1038)$:

$E(Y) = np = 10 \times 0.1038$

$E(Y) \approx 1.038$

(d) Find $P(Y = 3)$.

$P(Y = 3) = \binom{10}{3}(0.1038)^3(0.8962)^7$

$P(Y = 3) \approx 0.008$

(e) Find $P(24.15 < X < 25|X < 26)$.

$P(24.15 < X < 25|X < 26) = \frac{P(24.15 < X < 25)}{P(X < 26)}$

$P(X < 26) = 1 - 0.1038 = 0.8962$

$P(24.15 < X < 25|X < 26) = \frac{0.3554}{0.8962}$

$P(24.15 < X < 25|X < 26) \approx 0.3965$

Question 9

(a) Find the period of f.

Given $f(x) = x \sin\left(\frac{\pi}{3}x\right)$:
Period of $\sin\left(\frac{\pi}{3}x\right)$ is $\frac{6}{\pi} \times 2\pi = 6$.

(b) (i) Find the value of b.

Given local maximum at $(2, 21.8)$ and local minimum at $(8, 10.2)$:
$f(2) = 2 + 5\sin\left(\frac{\pi}{3}2 + b\right) = 21.8$
$f(8) = 8 + 5\sin\left(\frac{\pi}{3}8 + b\right) = 10.2$

Solving for b:

(ii) Find the value of $f(6)$.

Substitute $x = 6$ into f.

(c) Find the value of p and q.

Given points for g:
$g(3) = 2.5, \quad g(6) = 15.1$

Solve simultaneous equations.

(d) Find the value of x for which the functions have the greatest difference.

Find maximum of $|f(x) - g(x)|$.